Time-delay-induced instabilities in reaction-diffusion systems

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Time delay in the kinetic terms of reaction-diffusion systems has been investigated. It has been shown that short delay beyond a critical threshold may induce spatiotemporal instabilities. For unequal diffusivities and appropriate parameter space delay may induce Turing instability resulting in stationary patterns and also interesting Turing-Hopf transition with the formation of spirals. The theoretical scheme has been numerically explored in two different prototypical reaction-diffusion systems.

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I. INTRODUCTION

The use of ordinary partial differential equations in chemical kinetics is based on the idea that the reaction depends upon the current value of the variables present. A chemical event, however, may take a small but finite amount of time to affect a reaction at a later time. For the system which feels the effect at an appreciable long time later, the effect of delay cannot be ignored. Such types of systems are described by delay differential equations (DDEs). This delay often realized as feedback is quite pervasive in chemistry and biology [1,2] and may involve reaction term or diffusion term in a reaction-diffusion system. The present paper concerns time delay in the reaction terms of two-component reaction-diffusion systems.

The delay was first employed by Ott *et al.* [3] to control systems by manipulating the input signals adjusted to the current states of the systems. Delayed feedback and its modifications are widely used to control chaos and to stabilize unstable oscillations or steady states in spatially homogeneous systems [4–6]. Global delayed feedback can eliminate the defects and stabilize regular oscillations in fluids, and may result in suppression of turbulent states in plasma. Experimentally delayed feedback was used by Grill *et al.* [7], where the effect of sequence of short light impulses forcing a meandering spiral wave was investigated. The diameter of the synchronized trajectory of the spirals was increased as the delay between the registration of a wave front at the measuring point and triggering of the stimulus was increased and finally the synchronization broke down after a particular threshold value of delay. It has also been observed that electric field may induce an intrinsic delay in the system since the intensity of the local electric field depends on the concentration of ions at some former time [8]. Delay can modulate and induce complex dynamics in many systems including chemical reaction [9], DNA transcription, and optical systems. Hu *et al.* [10] showed that both traveling pattern (spatially periodic and oscillatory in time) and standing pattern (spatially periodic and stationary in time) are observed in the parametric domain, where undelayed system exhibits uniform oscillations.

Most of the above-mentioned studies deal with the operation of delayed feedback on the system in the pattern-forming region and explore how the existing spatiotemporal patterns could be modulated, stabilized, or suppressed by the delay. In earlier studies [11] the effect of time delay had been investigated in the context of photosensitive chlorine-dioxide-iodide-malonic acid (CDIMA) system, where the rate of photochemical reaction is affected by a local delay of the reaction component. The delayed feedback of the reactant on anharmonic oscillations [12] in the carbon monoxide oxidation reaction on a platinum single crystal surface has revealed interesting patterns representing traveling phase flips, dynamic clustering, and asynchronous oscillations. Time delay can also control the dynamic behavior of patterns in other issues [13]. Motivated by these observations our aim in this paper is to carry out a systematic search for instability conditions, which can lead to symmetry-breaking structures due to the time delay in the kinetic terms of a two-component reaction-diffusion system. Our objective here is to explore how delay plays a conspicuous role in determining the stability threshold of the steady state both in absence and presence of diffusion. Second it is also pertinent to enquire whether different regions of instability can be modulated by tuning the delay in kinetics. Our analysis reveals that for unequal diffusivities delay may induce Turing instability initiating stationary patterns and transition between Turing to Hopf instabilities resulting in the formation of spirals. We corroborate our theoretical analysis by numerical simulations on two specific reaction-diffusion systems [14–27].

II. GENERALIZED APPROACH TO DELAY INDUCED SPATIOTEMPORAL INSTABILITY

To start with we consider a reaction-diffusion system which describes the dynamics of two field variables $u(x,y,t)$ and $v(x,y,t)$, a function of space (x,y) and time (t). The delay is incorporated in the kinetic terms $f(u,v)$ and $g(u,v)$ as follows:

$$\frac{\partial u(x,y,t)}{\partial t} = f(u(x,y,t-\tau),v(x,y,t-\tau)) + d\frac{\partial^2 u}{\partial x^2} + d\frac{\partial^2 u}{\partial y^2}, \tag{2.1}$$

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$$\frac{\partial v(x,y,t)}{\partial t} = g(u(x,y,t-\tau),v(x,y,t-\tau)) + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}. \tag{2.2}$$

Here d is the ratio of diffusion coefficients of the two species: Assuming τ to be small we replace $u(x,y,t-\tau) = u(x,y,t) - \tau\frac{\partial u(x,y,t)}{\partial t}$ and $v(x,y,t-\tau) = v(x,y,t) - \tau\frac{\partial v(x,y,t)}{\partial t}$ in Eqs. (2.1) and (2.2) to write as

$$\frac{\partial u}{\partial t} = f\left(u(x,y,t) - \tau\frac{\partial u}{\partial t}, v(x,y,t) - \tau\frac{\partial v}{\partial t}\right) + d\frac{\partial^2 u}{\partial x^2} + d\frac{\partial^2 u}{\partial y^2} \tag{2.3}$$

and

$$\frac{\partial v}{\partial t} = g\left(u(x,y,t) - \tau\frac{\partial u}{\partial t}, v(x,y,t) - \tau\frac{\partial v}{\partial t}\right) + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}. \tag{2.4}$$

Expanding in Taylor series and neglecting the higher-order nonlinearities, Eqs. (2.3) and (2.4) become

$$\frac{\partial u}{\partial t} = f(u,v) - \tau f_u(u,v)\frac{\partial u}{\partial t} - \tau f_v(u,v)\frac{\partial v}{\partial t} + d\frac{\partial^2 u}{\partial x^2} + d\frac{\partial^2 u}{\partial y^2} \tag{2.5}$$

and

$$\frac{\partial v}{\partial t} = g(u,v) - \tau g_u(u,v)\frac{\partial u}{\partial t} - \tau g_v(u,v)\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}. \tag{2.6}$$

The homogenous steady states of the dynamical system are the fixed points u_0 and v_0 defined by

$$f(u_0,v_0) = 0, \quad g(u_0,v_0) = 0. \tag{2.7}$$

We now consider small spatiotemporal perturbations $\delta u(x,y,t)$ and $\delta v(x,y,t)$ on a homogenous steady state (u_0,v_0) so that we have

$$u(x,y,t) = u_0 + \delta u(x,y,t), v(x,y,t) = v_0 + \delta v(x,y,t). \tag{2.8}$$

By expanding the reaction terms around this steady state in a Taylor series up to first order we obtain

$$\frac{\partial \delta u}{\partial t} + \tau f_u\frac{\partial \delta u}{\delta t} + \tau f_v\frac{\partial \delta v}{\partial t} = f_u\delta u + f_v\delta v + d\frac{\partial^2 \delta u}{\partial x^2} + d\frac{\partial^2 \delta u}{\partial y^2} \tag{2.9}$$

and

$$\frac{\partial \delta v}{\partial t} + \tau g_u\frac{\partial \delta u}{\delta t} + \tau g_v\frac{\partial \delta v}{\partial t} = g_u\delta u + g_v\delta v + \frac{\partial^2 \delta v}{\partial x^2} + \frac{\partial^2 \delta v}{\partial y^2}. \tag{2.10}$$

Expressing spatiotemporal perturbation $\delta u(x,y,t)$ and $\delta v(x,y,t)$ in the form

$$\delta u(x,y,t) = \delta u_0 e^{\lambda t}\cos k_x x \cos k_y y, \tag{2.11}$$

$$\delta v(x,y,t) = \delta v_0 e^{\lambda t}\cos k_x x \cos k_y y, \tag{2.12}$$

and upon inserting them in Eqs. (2.9) and (2.10), we obtain the following matrix equation for eigenvalues:

$$\begin{pmatrix} \lambda(1+\tau f_u) - f_u + dk^2 & (\lambda\tau - 1)f_v \\ (\lambda\tau - 1)g_u & \lambda(1+\tau g_v) - g_v + k^2 \end{pmatrix}\begin{pmatrix} \delta u_0 \\ \delta v_0 \end{pmatrix} = 0. \tag{2.13}$$

Neglecting higher-order nonlinearities of $O(\tau^2)$ we get the following quadratic equation for the eigenvalues of the associated stability matrix:

$$\lambda^2 - \lambda\frac{[(f_u+g_v) + 2\tau(f_u g_v - g_u f_v)] - k^2\{(1+d) + \tau(f_u + dg_v)\}}{[1+(f_u+g_v)\tau]} + \frac{dk^4 - k^2(f_u+dg_v) + f_u g_v - g_u f_v}{[1+(f_u+g_v)\tau]} = 0, \tag{2.14}$$

where $k^2 = k_x^2 + k_y^2$. Our aim here is to find out the threshold or critical value of the delay time for which the delayed system which is otherwise stable with respect to homogeneous perturbation becomes unstable. For the homogeneous perturbation we have

$$\lambda^2 - \lambda\frac{[(f_u+g_v) + 2\tau(f_u g_v - g_u f_v)]}{[1+(f_u+g_v)\tau]} + \frac{f_u g_v - g_u f_v}{[1+(f_u+g_v)\tau]} = 0, \tag{2.15}$$

Eq. (2.15) may be written as

$$\lambda^2 - A\lambda + B = 0, \tag{2.16}$$

where $A = \frac{[(f_u+g_v)+2\tau(f_u g_v - g_u f_v)]}{[1+(f_u+g_v)\tau]}$ and $B = \frac{f_u g_v - g_u f_v}{[1+(f_u+g_v)\tau]}$.

The condition for stability of the homogeneous steady state for the system with delay is $A<0$ and $B>0$. B is positive if $1+(f_u+g_v)\tau>0$. Now the condition $A<0$ can be obtained when $[(f_u+g_v)+2\tau(f_u g_v - g_u f_v)]<0$ and $[1+(f_u+g_v)\tau]>0$. This allows the range of values for τ determined by the following condition:

$$-\frac{1}{f_u+g_v} < \tau < -\frac{f_u+g_v}{2(f_u g_v - g_u f_v)}. \tag{2.17}$$

We now introduce diffusion and enquire whether it gives rise to the situation of special interest in destabilizing of the homogeneous steady state of the system with delay when the ratio of the diffusion coefficients is unity ($d=1$). Putting $d=1$ in Eq. (2.14) and writing $C = \frac{[(f_u+g_v)+2\tau(f_u g_v - g_u f_v)] - k^2\{2+\tau(f_u+g_v)\}}{[1+(f_u+g_v)\tau]}$ and $D = \frac{k^4 - k^2(f_u+g_v) + f_u g_v - g_u f_v}{[1+(f_u+g_v)\tau]}$ we obtain

$$\lambda^2 - C\lambda + D = 0. \tag{2.18}$$

We now recall that the condition of stability of the homogeneous steady state ($f_u+g_v<0$ and $f_u g_v - g_u f_v>0$) asserts that D is always positive. Since the numerator of D determines the Turing line which is found to be independent of τ, any delay-induced instability in presence of diffusion must depend on C. Hence the condition of instability is $C>0$. Moreover since the denominator of C is always positive the condition of instability reduces to the following:

$$[(f_u + g_v) + 2\tau(f_u g_v - g_u f_v)] - k^2[2 + \tau(f_u + g_v)] > 0. \tag{2.19}$$

This implies that the lower bound of τ must satisfy

$$\tau > \frac{2k^2 - (f_u + g_v)}{2(f_u g_v - g_u f_v) - k^2(f_u + g_v)} = \tau_c. \tag{2.20}$$

The critical delay τ_c thus sets a condition of instability of the homogeneous steady state of the system. This is true even if the ratio of the diffusion coefficients is unity. However, in what follows from the detailed numerical simulations of the partial differential equations carried out in the next section, this instability does not give rise to any symmetry-breaking spatial structures.

When the diffusivities of the activator and the inhibitors are not equal ($d \neq 1$) then since D is always positive the necessary and sufficient condition for delay-induced instability is given by

$$\tau > \frac{(1+d)k^2 - (f_u + g_v)}{2(f_u g_v - g_u f_v) - k^2(f_u + g_v)} = \tau_c. \tag{2.21}$$

The conclusion is that the delay beyond a threshold may induce spatiotemporal instability in the two-component activator-inhibitor system when the undelayed system remains homogeneously stable. It also follows that the bifurcation lines can be manipulated by suitable application of short time delay in the kinetic terms to modulate the different regions of instability and induce transitions between them. We illustrate the theory with the help of the following two examples.

III. APPLICATIONS

A. Pigmentation fish model

As an example we first consider a two-variable reaction-diffusion system proposed by Barrio *et al.* [15] as an alternative approach to mechanochemical models, where pattern arises due to physical interaction between cells with external surrounding leading to cell aggregation and differentiation. The equations are given by

$$\frac{\partial u}{\partial t} = \alpha u(1 - r_1 v^2) + v(1 - r_2 u) + \delta d \frac{\partial^2 u}{\partial x^2}, \tag{3.1}$$

$$\frac{\partial v}{\partial t} = \beta v(1 + r_1 uv/\beta) + u(\gamma + r_2 v) + \delta \frac{\partial^2 v}{\partial x^2}, \tag{3.2}$$

where $\alpha, \beta, \gamma, r_1, r_2$ are given parameters of the dynamics. δ is the length scale. The motivation behind the choice of the reaction terms is the requirement of conservation of certain chemical species and nonlinearity, which determines the specific unstable modes to dominate for the selection of a typical pattern when Turing instability sets in. Since in the absence of diffusion the system admits one more solution at $v = -\frac{(\alpha+\gamma)u}{1+\beta}$, which follows simply from homogeneous steady state condition on Eqs. (3.1) and (3.2), the state (0,0) can be ensured as the only uniform steady state by setting the parameter $\alpha = -\gamma$. The complex patterns generated with this model under various conditions bear striking resemblance with pigmentation patterns observed in a number of fish species. For further details and other related work we refer to Refs [13–16,28].

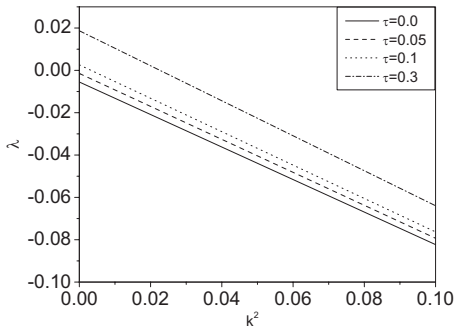

FIG. 1. Dispersion relation: Plot of λ vs k^2 for the parameter set α=0.899, β=−0.91, γ=−0.899, δ=2.0, r_1=0.02, and r_2=2.0 for different diffusivities of the two components (d=0.536).

The dynamical evolution of concentration of variables at a time depends on those at earlier time. We set the parameter values as α=0.899, β=−0.91, γ=−0.899, δ=2.0, r_1=0.02, and r_2=0.2 and depending on the ratio of the diffusivities two distinct situations (d=1 and $d \neq 1$) emerge. We first explore the instability threshold for $d \neq 1$ by plotting in Fig. 1 the dispersion relation, Re λ vs k^2 for several values of delay τ. It is observed that for very small values of delay the system remains homogeneous. The critical delay which can be calculated theoretically from Eq. (2.21) for the given set of parameter values and d=0.536 is found to be 0.37 [corresponding to a system size and number of nodes as illustrated in Fig. 2(b) subsequently in this section].

To realize the spatiotemporal instability and resulting patterns we have carried out numerical simulations of Eqs. (3.1)

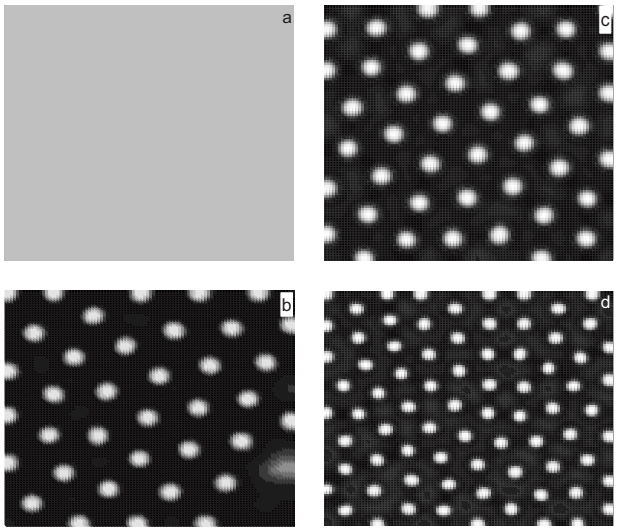

FIG. 2. Numerical simulations (in two-dimensional space) of delay-induced instability in pigmentation fish model for the parameters α=0.899, β=−0.91, γ=−0.899, δ=2.0, r_1=0.02, and r_2=0.2 (grid size 100×100 with $\Delta x = \Delta y = 1.0$, Δt=0.01) (a) d=0.536, τ=0.0 (homogeneous state), (b) d=0.536, τ=0.4 (delay-induced pattern), (c) d=0.516, τ=0.0 (Turing region), and (d) d=0.516, τ=0.4 (delay in Turing region).

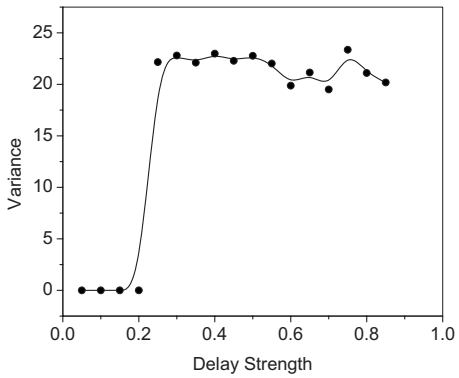

FIG. 3. Plot of variance vs delay strength for the parameter set $\alpha=0.899$, $\beta=-0.91$, $\gamma=-0.899$, $\delta=2.0$, $r_1=0.02$, and $r_2=2.0$ for different diffusivities of the two components ($d=0.536$).

and (3.2) in two dimensions using explicit Euler method. The computations have been performed on 100×100 array with grid spacing $\Delta x=\Delta y=1.0$ and a time step $\Delta t=0.01$ and zero flux boundary condition. The development of a typical time-delay-induced pattern in the form of spots for $d=0.536$ is shown in Figs. 2(a) and 2(b) for $\tau=0$ and $\tau=0.4$, respectively. In order to compare the theoretically calculated critical threshold for delay with the numerical one it is worthwhile to look into the behavior of an order parameter as a function of delay. The definition of this quantity (η) is $\eta=[\langle u^2\rangle-\langle u\rangle^2]/N^2$ (N is the number of sites in the grid). This order parameter has been used earlier in the context of pattern formation [28]. In Fig. 3 we display the numerically calculated variance for the aforesaid parameter values as a function of delay strength τ for $d=0.536$. The simulations are performed from a homogeneous state of the system to an inhomogeneous state of pattern, each state being considered after the system reaches its stationarity. It is therefore easy to locate the nonequilibrium transition point corresponding to an approximate critical value of delay strength of $\tau_c=0.3$. We observe a fair correspondence between theoretical and numerical values.

Next we consider the effect of delay in the Turing region. Figure 2(c) illustrates the typical Turing pattern in the form of spots for $d=0.516$ when $\tau=0$. It is apparent that with a delay $\tau=0.4$ the number of spots in the given two-dimensional domain increases, signifying an increase in the number of nodes in both directions [Fig. 2(d)]. This implies that delay may modify the length scale of the spatiotemporal dynamics in the Turing region. Lastly, we mention in passing that when the diffusivities are equal we do not observe any "bonafide" instability that entails a spatial structure, which can be quantitatively characterized by definite wavelength appropriate to system length scale.

B. Chlorine-dioxide-iodide-malonic acid system

The development of CDIMA system not only pioneered the experimental study in this field but also inspired extensive analytical and numerical studies that provided enormous insight into the subject [22–26]. The chemical reactions that involve five species malonic acid (MA), I_2, ClO_2, I^-, and ClO_2^- in the model are the following:

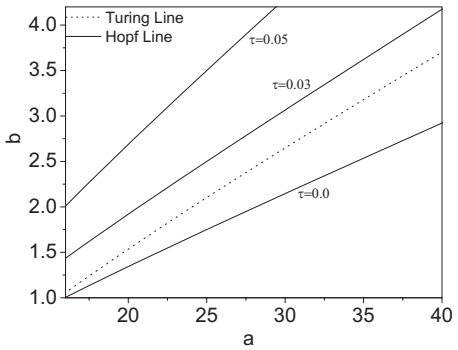

FIG. 4. Bifurcation diagram for CDIMA system exhibiting delay-induced Turing-Hopf transition for $d=1.6$, $\sigma=8.0$. Dashed line represents Turing line; solid line represents Hopf line.

$$\mathrm{MA}+\mathrm{I_2}\rightarrow\mathrm{IMA}+\mathrm{I^-}+\mathrm{H^+},$$

$$\mathrm{ClO_2}+\mathrm{I^-}\rightarrow\mathrm{ClO_2^-}+\frac{1}{2}\mathrm{I_2},$$

$$\mathrm{ClO_2^-}+4\mathrm{I^-}+4\mathrm{H^+}\rightarrow\mathrm{Cl^-}+2\mathrm{I_2}+2\mathrm{H_2O}.$$

The above-mentioned five-variable model was reduced by Lengyel and Epstein to a two-component system with the help of an experimentally realizable assumption that the concentrations of malonic acid, chlorine dioxide, and iodine remain practically constant. u and v can be identified as the dimensionless concentrations of activator (I^-) and inhibitor (ClO_2^-), respectively, for the reaction-diffusion system so that the equations become

$$\frac{\partial u}{\partial t}=a-u-\frac{4uv}{1+u^2}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}, \tag{3.3}$$

$$\frac{\partial v}{\partial t}=\sigma\left[b\left\{u-\frac{uv}{1+u^2}\right\}\right]+\sigma d\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]. \tag{3.4}$$

Here a, b, and σ are dimensionless parameters containing kinetic parameters and initial concentrations of the reactants with $\sigma=1+s_0\frac{k_1}{k_2}$. s_0 corresponds to the initial concentration of starch undergoing a complexation reaction with iodide to form starch iodide complex ($I^-+S\leftrightharpoons$complex). k_1 and k_2 are the forward and backward rate constants of the complexation equilibrium. The ratio of the diffusion coefficients d ($d=D_{ClO_2^-}/D_{I^-}$) of the activator and the inhibitor in absence of starch is related to the effective ratio of the diffusion coefficients δ for the dynamics as $\delta=\sigma d$. We fix the experimentally admissible parameter values as mentioned in Refs. [19–21]. The parameter σ is controlled by the concentration of starch.

It is well known from the linear stability analysis of the undelayed ($\tau=0$) system that by varying the complexing agent (σ), one can adjust the Hopf line in the b-a parameter plane in such a way that it lies below Turing bifurcation line, which is independent of σ (Fig. 4). Introduction of delay τ gives us a useful handle for further manipulation of the in-

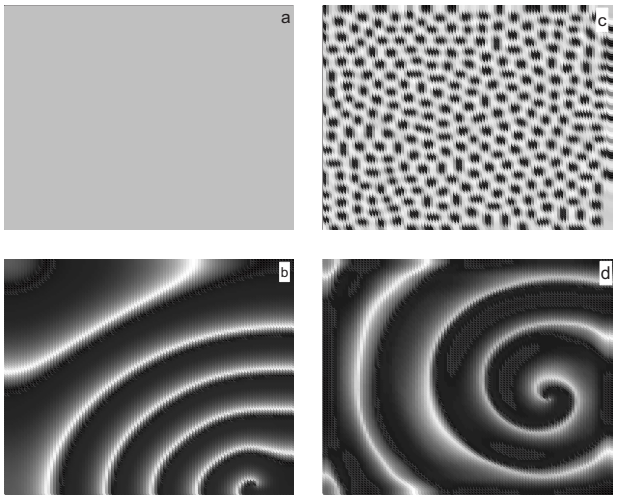

FIG. 5. Numerically simulated (in two-dimensional space) delay-induced spatial instability in CDIMA system for the parameters $a=18.0$, $b=1.5$, (grid size 100×100 with $\Delta x=\Delta y=1.0$ and $\Delta t=0.005$) (a) $\sigma=4.0$, $d=0.25$, $\tau=0.0$ (Hopf region), (b) $\sigma=4.0$, $d=0.25$, $\tau=0.2$ (delay in Hopf region), (c) $\sigma=8.0$, $d=1.6$, $\tau=0.0$ (Turing region), and (d) $\sigma=8.0$, $d=1.6$, $\tau=0.2$ (delay in Turing region).

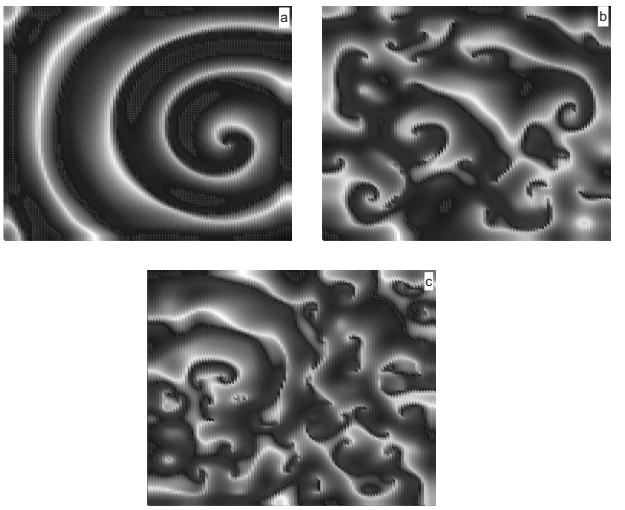

FIG. 6. Numerically simulated (in two-dimensional space) delay-induced deformation of spirals in CDIMA system for the parameters $a=18.0$, $b=1.5$, $\sigma=8$, $d=1.6$ (grid size 100×100 with $\Delta x=\Delta y=1.0$ and $\Delta t=0.005$) (a) $\tau=0.2$, (b) $\tau=0.21$, and (c) $\tau=0.225$.

stability region between Hopf and Turing lines, since it can be easily shown that the Hopf bifurcation line for the system with delay ($\tau\neq0$) is given by

$$b=\frac{3a^2-125}{5\sigma a(1-10\tau)}, \tag{3.5}$$

whereas the Turing line remains unaffected by the delay which may be written as

$$(3da^2-5ab-125d)^2=100abd(a^2+125). \tag{3.6}$$

Hopf line can be suitably adjusted in the a-b plane by the appropriate choice of delay so that it crosses the Turing line. By increasing the delay the width of the Hopf region can be widened. This is shown in Fig. 4. It is thus apparent that by tuning the delay the Hopf bifurcation line can be switched over to the Turing line, resulting in a condition for limit cycle oscillations. Furthermore when diffusion comes into play in such a system where the reaction kinetics alone exhibits periodic limit cycle behavior via a Hopf bifurcation, one may envisage periodic wave train solutions. Such systems have characteristic properties similar to well-known $\lambda-\omega$ systems in the vicinity of Hopf bifurcation, which gives rise to spirals [29–31]. We therefore expect that the delay may induce spirals via a Turing-Hopf transition.

In view of the aforesaid analysis we carry out numerical simulations using Eqs. (3.3) and (3.4) by the explicit Euler method as described earlier in search of Turing-Hopf transition induced by time delay. The results are illustrated in Fig. 5. The simulations are started with spatially random perturbations of $\sim1\%$ around the steady state $u_0=a/5$, $v_0=1+u_0^2$ and for $\sigma=4$, which is a Hopf region. The system without delay remains homogeneous in this parameter regime. Application of delay beyond a critical threshold exhibits a spiral [Fig. 5(b)] pattern. When σ is raised to a higher value the system returns to usual Turing region [$\sigma=8.0$, Fig. 5(c)] for $d=1.6$ and beyond a critical delay strength the spots get deformed and produce a regular spiral [Fig. 5(d)]. This clearly demonstrates a Turing-Hopf transition induced by delay. Once the spiral is generated in the Turing region as a result of Turing-Hopf transition, a systematic small increase in the delay leads to its breakup along with the formation of small deformed spirals. This is shown in Figs. 6(a)–6(c). Again for equal diffusivities we do not observe any regular structures as in the earlier cases.

IV. CONCLUSION

In summary, we have formulated a generalized approach to time delay-induced instabilities of the homogeneous steady state of the two-component reaction-diffusion systems. It has been shown that the instabilities can be realized when the diffusivities of the two components are different. An interesting offshoot of the theoretical scheme is a critical delay which determines the threshold of instability. We have demonstrated that delay in kinetics can be instrumental in inducing Turing instability as well as Turing-Hopf transition when the diffusion ratio differs from unity. This may result in the formation of symmetry-breaking structures in the form of Turing patterns or spirals. The delay in reaction kinetic terms may thus be tuned to manipulate the bifurcation lines and the associated stability regions between them. Since time delay appears as an important component of the dynamical systems in several areas of biology, we believe that our observation of delay-induced instability is likely to be important in several far-from-equilibrium phenomena.

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[1] F. R. Sharpe and A. J. Lotka, Am. J. Trop. Med. Hyg. **3**, 96 (1923).
[2] K. W. Smith and R. M. Noyes, J. Phys. Chem. **87**, 1520 (1983).
[3] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[4] M. E. Bleich and J. E. S. Socolar, Phys. Lett. A **210**, 87 (1996).
[5] A. G. Balanov, N. B. Janson, and E. Scholl, Phys. Rev. E **71**, 016222 (2005).
[6] P. Hovel and E. Scholl, Phys. Rev. E **72**, 046203 (2005).
[7] S. Grill, V. S. Zykov, and S. C. Muller, Phys. Rev. Lett. **75**, 3368 (1995).
[8] S. Dutta and D. S. Ray, Phys. Rev. E **77**, 036202 (2008).
[9] M. J. Piotrowska, Math. Comput. Modell. **42**, 123 (2005).
[10] H. X. Hu, Q. S. Li, and S. Li, Chem. Phys. Lett. **447**, 364 (2007).
[11] L. Ji and Q. S. Li, Chem. Phys. Lett. **391**, 176 (2004).
[12] M. Bertram and A. S. Mikhailov, Phys. Rev. E **63**, 066102 (2001).
[13] T. Sakurai, E. Mikhailov, F. Chirila, and K. Showalter, Science **296**, 2009 (2002).
[14] A. M. Turing, Philos. Trans. R. Soc. London **327**, 37 (1952).
[15] R. A. Barrio, C. Verea, J. Aragon, and P. K. Maini, Bull. Math. Biol. **61**, 483 (1999).
[16] S. S. Riaz, S. Kar, and D. S. Ray, J. Chem. Phys. **121**, 5395 (2004).
[17] S. S. Riaz, R. Sharma, S. P. Bhattecharyya, and D. S. Ray, J. Chem. Phys. **127**, 064503 (2007).
[18] R. T. Liu, S. S. Liaw, and P. K. Maini, Phys. Rev. E **74**, 011914 (2006).
[19] R. A. Barrio, R. E. Baker, B. Vaughan, K. Tribuzy, M. R. de Carvalho, R. Bassanezi, and P. K. Maini, Phys. Rev. E **79**, 031908 (2009).
[20] P. De Keeper, J. Boissonade, and I. R. Epstein, J. Phys. Chem. **94**, 6525 (1990).
[21] P. K. Maini, C. R. Biol. **327**, 225 (2004).
[22] V. Castets, E. Dulos, J. Boissonade, and P. De Kepper, Phys. Rev. Lett. **64**, 2953 (1990).
[23] P. De Kepper, V. Castets, E. Dulos, and J. Boissonade, Physica D **49**, 161 (1991).
[24] I. Lengyel and I. R. Epstein, Science **251**, 650 (1991).
[25] S. S. Riaz, S. Kar, and D. S. Ray, Physica D **203**, 224 (2005).
[26] S. S. Riaz, S. Banerjee, S. Kar, and D. S. Ray, Eur. Phys. J. B **53**, 509 (2006).
[27] S. Dutta, S. S. Riaz, and D. S. Ray, Phys. Rev. E **71**, 036216 (2005).
[28] S. S. Riaz, S. Kar, S. Dutta, and D. S. Ray, Eur. Phys. J. B **47**, 255 (2005).
[29] J. D. Murray, *Mathematical Biology* (Springer-Verlag, Berlin, 1993).
[30] M. R. Duffy, N. F. Britton, and J. D. Murray, SIAM J. Appl. Math. **39**, 8 (1980).
[31] Y. Xu-Jin, S. Xin, L. Hui-Min, and O. Qi, Chin. Phys. Lett. **26**, 024702 (2009).